1. Great progress on T35 and SS4 – you were very close!

2. I think we know what we need to prove to resolve P31, T28, T30, T31 and T32. Don’t give up on these, but don’t spend all your time on them.


4. Start working on the rest of the integration chapter and basic set theory as well. I know I’m opening up the flood gates, but that’s ok. I really wanted us to work hard on some of these hard problems early on to push you. Now I want to work a bit more on breadth than depth!

5. Keep up the good work!

**Definition 1** The statement that \(q_1, q_2, q_3, \ldots\) is a subsequence of \(p_1, p_2, p_3, \ldots\) means that there is an increasing sequence of natural numbers, \(n_1, n_2, n_3, \ldots\) such that for each natural number \(i\), we have \(p_{n_i} = q_i\).

**Example:** Suppose \(p_1, p_2, p_3, \ldots\) is a sequence and \(n\) is a function with domain the natural numbers so that \(n_1 = 2, n_2 = 4, n_3 = 6, \ldots\) so that \(n(k) = 2k\) for \(k = 1, 2, 3, \ldots\). Then the sequence \(n\) defines the sequence \(q_1 = p_2, q_2 = p_4, q_3 = p_6, \ldots\) which is a subsequence of \(p_1, p_2, p_3, \ldots\). We will use the notation \((p_{n_k})\) for the sequence \(p_1, p_2, p_3, \ldots\) and \((p_{n_k})\) for the subsequence of \(p\) defined by the sequence \(n_1, n_2, n_3, \ldots\). Notice that for any sequence \(n\) defining a subsequence, \(n_k \geq k\) because \(n\) is an increasing sequence.

**SS 0** Prove that the subsequence in the above example converges.

**SS 0’** Prove that the subsequence in the above example converges as long as \(p_1, p_2, \ldots\) converges!

Thanks to Kimberly for the nice counter example that showed that SS0 is false. To remedy this, I’ve created what I intended, which is SS0’. As someone in class pointed out (Jessica, I think) since Tre resolved Kimberly’s question below, we know that SS0’ is true and don’t need to prove it.

**SS 1 (Kimberly’s Question)** If \(q_1, q_2, q_3, \ldots\) is a subsequence of \(p_1, p_2, p_3, \ldots\) and \(p_1, p_2, p_3, \ldots\) converges to some number \(x\), then must \(q_1, q_2, q_3, \ldots\) converge to \(x\)?

Now that Kimberly’s Question is resolved by Tre, we have the theorem:

**Theorem 1** If \(q_1, q_2, q_3, \ldots\) is a subsequence of \(p_1, p_2, p_3, \ldots\) and \(p_1, p_2, p_3, \ldots\) converges to some number \(x\), then \(q_1, q_2, q_3, \ldots\) also converges to \(x\).

**SS 2 (John)** Suppose that \(q_1, q_2, q_3, \ldots\) is a subsequence of \(p_1, p_2, p_3, \ldots\) and there is a number \(x\) so that \(q_1, q_2, q_3, \ldots\) converges to \(x\). Is it true that \(p_1, p_2, p_3, \ldots\) converges to \(x\)?

John resolved this by an example: \(p_n = (-1)^n\) is a sequence and has a convergent subsequence, but the original sequence does not converge. Notice that this also resolves SS3.

**SS 3** Suppose that \((p_n)_{n=1}^{\infty}\) is a sequence of points in the closed interval \([a, b]\). Is it true that every subsequence of \((p_n)_{n=1}^{\infty}\) converges to some point in \([a, b]\)?

Since John resolved this, it now leads to the question SS5 below – must SOME subsequence converge? This leads us to SS5 below.

**Definition 2** The statement that the sequence \(p_1, p_2, p_3, \ldots\) is a Cauchy sequence means that if \(\epsilon\) is a positive number, then there is a positive integer \(N\) such that if \(n\) is a positive integer and \(m\) is a positive integer, \(n \geq N\), and \(m \geq N\), then the distance from \(p_n\) to \(p_m\) is less than \(\epsilon\).
SS4 The sequence \( p_1, p_2, p_3, \ldots \) is a Cauchy sequence if and only if it is true that for each positive number \( \epsilon \), there is a positive integer \( N \) such that if \( n \) is a positive integer and \( n \geq N \), then \( |p_n - p_N| < \epsilon \).

SS5 Suppose that \( (p_n)_{n=1}^{\infty} \) is a sequence of points in the closed interval \([a, b]\). Show that some subsequence of \( (p_n)_{n=1}^{\infty} \) converges to some point in \([a, b]\).

SS6 If the sequence \( p_1, p_2, p_3, \ldots \) converges to a point \( x \), then \( p_1, p_2, p_3, \ldots \) is a Cauchy sequence.

SS7 If \( p_1, p_2, p_3, \ldots \) is a Cauchy sequence, then the set \( \{p_1, p_2, p_3, \ldots\} \) is bounded.

SS8 If \( p_1, p_2, p_3, \ldots \) is a Cauchy sequence, then the set \( \{p_1, p_2, p_3, \ldots\} \) does not have two limit points.

SS9 If \( p_1, p_2, p_3, \ldots \) is a Cauchy sequence, then the sequence \( p_1, p_2, p_3, \ldots \) converges to some point.

Definition 3 An open cover of a set \( M \) is a collection of open intervals with the property that if \( m \in M \) then \( m \) is in at least one of the open intervals in the collection.

Definition 4 A set \( M \) is \( D1 \)-Compact if every open cover of \( M \) has a finite subcover.

Definition 5 A set \( M \) is \( D2 \)-Compact if every sequence in \( M \) has a convergent subsequence that converges to some point of \( M \).

Problem C1 Show that if \( M \) is a set that is \( D1 \)-compact then it is \( D2 \)-compact.

To completely resolve the nice start on T32 that Shaymal made, we need to resolve to things: SS5 and the following Lemma.

Lemma for T32 If \( f \) is continuous on \([a, b]\) and \((x_n)\) is a sequence in \([a, b]\) satisfying

1. \( f(x_1) > 1 \) and
2. \( f(x_n) > n \) and \( f(x_n) > f(x_{n-1}) \) for all \( n > 2 \)

then no subsequence of \((f(x_n))\) can converge.