We are very close to some nice results!

1. You are almost finished showing that every Cauchy Sequence is a convergent sequence – only SS9 stands in our way.

2. T32 and T33 will show us that the continuous image of a closed interval is either a point or a closed interval since we will know the range of a continuous function is both closed and bounded.

3. And of course the last two problems of the integration chapter will give us the Fundamental Theorem of Calculus!

4. The first two measure theory problems will classify all bounded open sets – every bounded open set is just a countable union of open intervals!

5. And soon we will know more about what sets are countable also.

A note on measure theory...

The second problem Theorem 91, should really be three problems! If \( O \) is a bounded open set, and \( G \) is the set of all components of \( O \), then

a. the union of all the components of \( O \) is \( O \)

b. the components of \( O \) are mutually disjoint (i.e. if \( p \) and \( q \) are distinct elements of \( O \) then the component of \( O \) containing \( p \) is either equal to the component of \( O \) containing \( q \) or is equal to it.)

c. the collection \( G \) is countable

A bit on countability...

**Definition 1** The set \( M \) of real numbers is countable if there is a sequence, \( x_1, x_2, \ldots \) with range \( M \).

**CNT1** Are the integers countable? Either show that there is a sequence with range the integers or show that no sequence can have range the integers.

**CNT2** Suppose that \( M_1 \) and \( M_2 \) are two countable sets. Show that \( M_1 \cup M_2 \) is a countable set.

**CNT3** Suppose that \( M_1, M_2, \ldots \) are countable sets. Show that \( M_1 \cup M_2 \cup M_3 \ldots \) is a countable set.

**Definition 2** A function \( f \) is uniformly continuous on the set \( M \) if for every \( \epsilon > 0 \) there exists a number \( \delta > 0 \) so that if \( u, v \in M \) and \( |u - v| < \delta \) then \( |f(u) - f(v)| < \epsilon \).

**Definition 3** A function \( f \) is continuous at the point \( p \) in \( M \) if for every \( \epsilon > 0 \) there exists a number \( \delta > 0 \) so that if \( x \in M \) with \( |x - p| < \delta \) then \( |f(x) - f(p)| < \epsilon \).

**Definition 4** A function \( f \) is continuous on the set \( M \) if it is continuous at each point in \( M \).

**UC1** Show that if \( f \) is continuous at each point \( x \in [a, b] \) then \( f \) is uniformly continuous on \( [a, b] \).

**UC2** Show that if \( f \) is uniformly continuous on \( [a, b] \) then \( f \) is continuous at each point \( x \in [a, b] \).

**UC3** Show that there is a set \( M \) of the reals and a function \( f \) defined on \( M \) so that \( f \) is continuous on \( M \) but \( f \) is not uniformly continuous on \( M \).
Definition 5 The statement that $q_1, q_2, q_3, \ldots$ is a subsequence of $p_1, p_2, p_3, \ldots$ means that there is an increasing sequence of natural numbers, $n_1, n_2, n_3, \ldots$ such that for each natural number $i$, we have $p_{n_i} = q_i$.

Example: Suppose $p_1, p_2, p_3, \ldots$ is a sequence and $n$ is a function with domain the natural numbers so that $n_1 = 2, n_2 = 4, n_3 = 6, \ldots$ so that $n(k) = 2k$ for $k = 1, 2, 3, \ldots$. Then the sequence $n$ defines the sequence $q_1 = p_2, q_2 = p_4, q_3 = p_6, \ldots$ which is a subsequence of $p_1, p_2, p_3, \ldots$. We will use the notation $(p_n)$ for the sequence $p_1, p_2, p_3, \ldots$ and $(p_{n_k})$ for the subsequence of $p$ defined by the sequence $n_1, n_2, n_3, \ldots$. Notice that for any sequence $n$ defining a subsequence, $n_k \geq k$ because $n$ is an increasing sequence.

SS 0 Prove that the subsequence in the above example converges.

SS 1 (Kimberly’s Question) If $q_1, q_2, q_3, \ldots$ is a subsequence of $p_1, p_2, p_3, \ldots$ and $p_1, p_2, p_3, \ldots$ converges to some number $x$, then must $q_1, q_2, q_3, \ldots$ converge to $x$?

Now that Kimberly’s Question is resolved by Tre, we have the theorem:

Theorem 1 If $q_1, q_2, q_3, \ldots$ is a subsequence of $p_1, p_2, p_3, \ldots$ and $p_1, p_2, p_3, \ldots$ converges to some number $x$, then $q_1, q_2, q_3, \ldots$ also converges to $x$.

SS 2 (John) Suppose that $q_1, q_2, q_3, \ldots$ is a subsequence of $p_1, p_2, p_3, \ldots$ and there is a number $x$ so that $q_1, q_2, q_3, \ldots$ converges to $x$. Is it true that $p_1, p_2, p_3, \ldots$ converges to $x$?

John resolved this by an example: $p_n = (-1)^n$ is a sequence and has a convergent subsequence, but the original sequence does not converge. Notice that this also resolves SS3.

SS 3 Suppose that $(p_n)_{n=1}^{\infty}$ is a sequence of points in the closed interval $[a, b]$. Is it true that every subsequence of $(p_n)_{n=1}^{\infty}$ converges to some point in $[a, b]$?

Since John resolved this, it now leads to the question SS5 below – must SOME subsequence converge?

Definition 6 The statement that the sequence $p_1, p_2, p_3, \ldots$ is a Cauchy sequence means that if $\epsilon$ is a positive number, then there is a positive integer $N$ such that if $n$ is a positive integer and $m$ is a positive integer, $n, m \geq N$, then the distance from $p_n$ to $p_m$ is less than $\epsilon$.

SS 4 The sequence $p_1, p_2, p_3, \ldots$ is a Cauchy sequence if and only if it is true that for each positive number $\epsilon$, there is a positive integer $N$ such that if $n$ is a positive integer and $n \geq N$, then $|p_n - p_N| < \epsilon$.

SS 5 Suppose that $(p_n)_{n=1}^{\infty}$ is a sequence of points in the closed interval $[a, b]$. Show that some subsequence of $(p_n)_{n=1}^{\infty}$ converges to some point in $[a, b]$.

SS 6 If the sequence $p_1, p_2, p_3, \ldots$ converges to a point $x$, then $p_1, p_2, p_3, \ldots$ is a Cauchy sequence.

SS 7 If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the set $\{p_1, p_2, p_3, \ldots\}$ is bounded.

SS 8 If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the set $\{p_1, p_2, p_3, \ldots\}$ does not have two limit points.

SS 9 If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the sequence $p_1, p_2, p_3, \ldots$ converges to some point.

Definition 7 An open cover of a set $M$ is a collection of open intervals with the property that if $m \in M$ then $m$ is in at least one of the open intervals in the collection.

Definition 8 A set $M$ is D1-Compact if every open cover of $M$ has a finite subcover.

Definition 9 A set $M$ is D2-Compact if every sequence in $M$ has a convergent subsequence that converges to some point of $M$.

Problem C1 Show that if $M$ is a set that is D1-Compact then it is D2-Compact.