The Teaching of Geometry

David M. Clark
March 2016

Geometry as we now understand it was first organized into a systematic whole by Euclid, the great scholar of Alexandria, who wrote his *Elements of Geometry* [7] around the year 300 B.C. According to the historian Howard Eves [8], "No work, except The Bible, has been more widely used. . . ." The *Elements* became a central pillar of western thought, studied by everyone seeking a broad education for well over two millennia.

The launch of Sputnik in 1957 awakened an urgent concern that the United States was being overtaken in science and mathematics by the Soviet Union. Part of the response to this concern was the commissioning of the School Mathematics Study Group (SMSG), financed by the National Science Foundation from 1958 to 1977, to create a reformed curriculum for school mathematics that came to be known as the "New Math". As it turned out, the SMSG decided not to include Euclid's axiomatic development of geometry in the New Math and it has not since been a part of our high school geometry curriculum.

As an informal assessment of this change, I have asked a variety of educated adults to tell me what they remember of high school geometry. Most of those who had taken geometry prior to about 1975 lit up and told me how it stood out for them as the time they realized that mathematics is not just about carefully following rules. It is a system of thought in which they can figure out what is true on their own, without needing to rely on books, teachers and other authorities. For them high school geometry was truly memorable. Most of those who took geometry after 1980 had little memory of the experience and could only tell me that they had studied something about triangles and circles. I had to conclude that, for them, it was of very little value.

From these testimonies it is clear to me that something important was lost from Euclid's geometry with the New Math curriculum. Since this change was made we have seen a series of revisions and failures of our school mathematics curricula. The Common Core State Standards were widely adopted, but states are now, one by one, abandoning them.

I will argue here that we should retain the CCSS as guidelines for what students need to learn from high school mathematics, but we must recognize that these guidelines represent a drastic change from what has been taught in geometry since 1980. As a result the present generation of teachers are largely products of that system, and implementation of the CCSS for geometry cannot be successful until our teachers themselves gain a modern mastery of the subject that is consistent with these standards. This will require serious professional development for current teachers, perhaps through programs like Math Circles, and revised preparation of preservice teachers at colleges and universities.

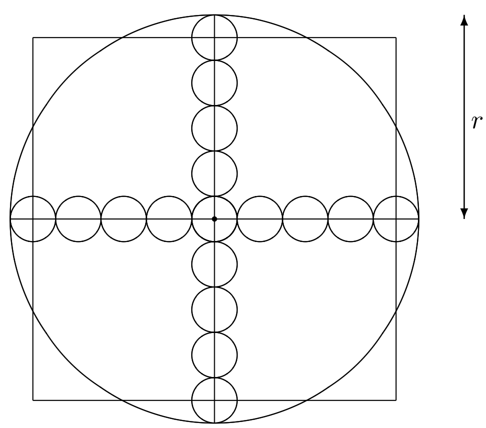
I will give here four central goals for this revised preparation that are drawn from the lessons of history. I will then examine the widely used systems of geometry to show that no one of them meets all four goals. Finally I will describe a new system that offers a means of resolving this ongoing dilemma.

1. Axiomatic Development

In order to understand the importance of Euclid's contribution, we must ask ourselves why it was that, for so many centuries, Euclidean geometry was viewed as an essential component of a sound education. Not just for aspiring mathematicians, scientists and engineers, but for everyone. To answer this question we need to go back in time and review the history of geometry.

Evidence of geometric principles are found in relics of the earliest civilizations that needed to construct structures that would stand and to measure fields to grow their food. They date back as much as five thousand years to ancient India, Babylon, Egypt and perhaps elsewhere. Prior to the classical Greek period the facts of geometry were established by conducting experiments, making conjectures and keeping those that seemed valid while discarding those that did not.

An excellent example of such a "fact" was found on the famous Rhind Papyrus from Egypt, dated around 1650 B.C. This document includes a straight edge and compass method to take a given circle and construct a square having the same area. This method is believed to have originated with the Babylonians and to have continued in use for a number of centuries. The construction is illustrated below. Starting at the center of the circle, construct two perpendicular diameters. Then divide each diameter into nine congruent segments, as illustrated by the nine circles on each. Finally construct a perpendicular to each diameter at the centers of the two circles at either end. These four perpendiculars form the required square.



Babylonian method to square a circle.

A careful examination of this construction reveals that the area of the square is very close to that of the circle, but just slightly bigger. This discrepancy was well within the error tolerance of the needs of the time, and much too small to be detected by the tools available. What is significant about this construction is that it was part of a body of knowledge that was transmitted by citing recognized authorities rather than by giving a rationale as to why it was valid. Indeed, there could be no rationale since the conclusion was in fact false. In this respect it was typical of much of the mathematical knowledge of the time.

The significance of Euclid's *Elements* lies not only in the practical value of the geometric knowledge it provides, but more importantly in its ability to establish that knowledge without the need to cite any external authority. Instead, Euclid presented a small list of facts, called *axioms*, which he asked his readers to experimentally verify as thoroughly as they were able. By logical reasoning, he then argued that an amazingly complex collection of other facts could be deduced from those axioms alone. Provided the axioms were all true, we could have full confidence that the conclusions drawn from them were true as well.

Euclid of Alexandria, 300 B.C.

This method of gaining reliable knowledge has had an enormous influence on the development of Western thought. It was reflected and reinforced in 1687 when Isaac Newton published his *Principia*. In this three volume work Newton laid out his laws of physics; a small set of axioms from which he deduced the entire theory of classical mechanics while inventing the calculus along the way. In a quite different domain, Thomas Jefferson justified the Declaration of Independence by carefully laying out his assumptions and then deducing from them their inevitable consequence.

> *"We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights,. . . "*

When Abraham Lincoln first ran for Congress, he took up a study of Euclid's *Elements*. In his famous 1859 debate with his rival Stephen Douglas over the legitimacy of slavery, he cited Euclid's central point:

> *"There are two ways of establishing a proposition. One is by trying to demonstrate it upon reason, and the other is, to show that great men in former times have thought so and so, and thus to pass it by the weight of pure authority."*

He went on to admonish Judge Douglas to demonstrate the legitimacy of slavery through a reasoned argument "*as Euclid demonstrated his propositions*," rather than to justify it by citing the authority of previous slave owners. Albert Einstein's relativity theory deduces a myriad of unintuitive facts from the single axiom that the speed of light is *not* relative to the observer.

As the above examples show, Euclidean geometry offered an exact system of assumptions and deductions that served as a model for less exacting areas of study. Because of its historically demonstrated power to establish reliable truth without citing previously accepted authorities, the axiomatic method of Euclid has been a central pillar of western thought for over two millennia. I will therefore list it as a first goal for a geometry curriculum.

- *Axiomatic Development.* Present geometry through an axiomatic development that begins with a small set of intuitive axioms from which the entire subject is derived.

2. Guided Inquiry

In the early twentieth century, R. L. Moore at the University of Texas made the observation that Euclid's method could bring us even more. It is surely an inspiration, he reasoned, to see how great thinkers like Newton, Jefferson, Lincoln and Einstein could give us important new insights by following the paradigm of Euclid. Moore's observation was that, if teaching were done right, we would not just be *demonstrating* that paradigm to our students. We would be *imparting* it to them so that they could use it themselves.

For example, suppose students read Euclid's proof that triangles with congruent angles are similar, or it is presented to them in a lecture, and they follow each line and agree that it is correct. They will then need to agree that the conclusion is correct as well. But they may have no idea how the proof was discovered, and they will probably not be able to reproduce it. In particular, they will have no confidence that they could find a similar proof of a theorem that they needed to prove themselves.


Robert L. Moore

Moore maintained that students receiving their mathematics course work in this way have been very poorly served. This kind of demonstration that a theorem is true is, for them, a small step beyond telling them that it has been previously accepted by appropriate authorities. If we want to give students access to mathematical knowledge, we must find ways to impart Euclid's axiomatic method to them as a skill that they can use to establish knowledge themselves whenever they need it. This skill would empower them with the ability to determine what is and is not true well beyond geometry, and even beyond mathematics.

In 1905 Moore completed his PhD, writing a dissertation on the axioms of plane geometry. In his own teaching in Texas he sought to eliminate traditional lectures and put students in charge of doing and presenting the work. Using what became known as the *Moore Method*, he provided his students with definitions and axioms, together with problems to solve and theorems to prove, and left it to them solve the problems and prove the theorems on their own. Class time was primarily spent with individual students presenting their work with him as a moderator and guide. For an excellent account of the use of the Moore Method see the Coppin, Mahavier, May and Parker [5]. Over time Moore, his colleagues, his PhD students and his collective 1,678 descendants [18] have modified his method to adopt it to the wide range of learning environments we have today.

A similar line of development has happened in other places and other disciplines. The common insight is that the proper role of a teacher is not to demonstrate his or her ability to use the methods of the discipline to gain knowledge, but rather to impart those methods to the students. This is done by eliminating lectures and using class time to engage the students in some form of active learning. In the 1970s Clarence Stephens at SUNY Potsdam created in this way the "Potsdam Miracle" in which as much as 10% of the students at this small liberal arts school were graduating with a degree in mathematics [20]. Nobel laureate Carl Wieman spent years developing a form of active learning for large sections of physics [6]. David Hanson founded the NSF funded and widely used program POGIL (Process Oriented Guided Inquiry Learning) for teaching chemistry with students actively working in small groups. Paulo Freire's 1968 book *Pedagogy of the Oppressed* [10] has led to the growing movement toward active learning in the social sciences, referred to as "critical pedagogy". In mathematics active learning is called "inquiry-based learning" or "guided inquiry learning". The "Moore Method" is the form of guided inquiry learning generally used in more advanced university courses.

In recent years there has been a growing consensus in the educational research community, and particularly within the STEM disciplines, around the value of active learning pedagogies. A 2014 article in the Proceedings of the National Academy of Sciences [9] reported on a metastudy of 225 studies of the relative benefits to students of lecture and active learning courses. Based on the statistical data they found, they concluded that the case in favor of active learning over lecture was so strong that, had these been medical studies revealing the same statistics, many of them would have qualified for being "stopped for benefit". That is, it would have be suggested to halt the study in order to avoid withholding the trial treatment (active learning) from the control group (lecture).

A three year study of guided inquiry learning in mathematics was headed by Sandra Laursen as her group witnessed it practiced by the Universities of California at Santa Barbara, Chicago, Michigan and Texas. They concluded that it "benefited students in multiple, profound, and perhaps lasting ways. Learning gains and attitudinal changes were especially positive for groups that are often under-served by traditional lecture-based approaches, including women and lower-achieving students." [16] and that "Evidence for increased persistence is seen among the high-achieving students ..." [15]

Students working themselves through the logical development of Euclidean geometry discovered how conclusions follow from given information, a skill they would need in all walks of life. It fostered the growth of critical thinking, an attribute required of the citizens of any functioning democracy. For these reasons I have added guided inquiry as a second goal for a college geometry curriculum.

- *Guided Inquiry.* Provide a presentation with minimal explanations and proofs that leads students to solve problems and prove theorems on their own while the instructor serves as a mentor and a guide.

3. Problems with Euclid

As mathematics began to mature after the Renaissance, mathematicians looked more carefully at Euclid's development and discovered that it failed to meet his stated objectives in three ways - each of which required fundamentally new advances in mathematics in order to rectify.

First, the proofs of his theorems contained hidden assumptions that were not explicitly stated in his axioms. For centuries mathematicians attempted to resolve this problem by filling in the missing assumptions. This proved to be a devilishly difficult task, as it was unclear exactly what constituted a correct mathematical proof and what we already knew about physical space that we could use to create a theory describing it. Examples of this fall into three categories: incidence, betweenness and intersections. Here is an example of each. We write "$[XYZ]$" to mean point Y is between points X and Z.

Incidence: *Not all points lie on (are incident to) one line.*

Betweenness: *If $[ABC]$ and $[ACD]$, then $[ABD]$.*

Intersections: *A line containing a point inside a circle intersects that circle.*

Euclid made many similar unstated assumptions in each of these categories. The question was this. Are these facts about physical space that we already know, or do they need to be proven? Mathematicians began to argue that we needed to prove them by finding clever proofs of absurd conclusions (like all triangles are isosceles) obtained by ignoring these issues.

Secondly, there was general agreement that Euclid's axioms themselves were assertions that we know to be true from experimental evidence with one exception. This was Euclid's Parallel Postulate: *For every line ℓ and point P not on ℓ, there is at most one line containing P that is parallel to ℓ.* Euclid himself was conservative in the use of this axiom, which he avoided as long as possible. For centuries mathematicians attempted to justify the Parallel Postulate by proving it from Euclid's other axioms, but these efforts remained fruitless.

Finally "geometry", from "earth measure" in Greek, was intended to refer to a system of numerical measurement of physical objects. Yet the early Greeks had access to only the rational number system, and had no effective means to describe the area or volume of an irregular shape. They knew from Pythagoras's theorem that there were many segments that did not have a rational length. This fact led Euclid to favor an axiomatic formulation of geometry over a numerical formulation. The later books of the *Elements* illustrate Euclid's awkward struggle with segments whose lengths are not in a rational ratio. These shortcomings of Euclid point to a third goal for a geometry curriculum.

- *Mathematical Foundations.* While we may not include the full mathematical foundations in the curriculum itself, they should be understood and available to the instructors who teach it.

The late nineteenth century saw a number of fundamental advances in mathematics that finally led to an understanding of these foundations. Boole, Frege and others began to clarify and formalize set theory and logic, which underly all of mathematics and help to delineate what constitutes a mathematical proof. Beltrami, Klein and Poincaré finally demonstrated that the desired proof of the Parallel Postulate was impossible if Euclidean geometry was consistent. They did this by showing that any model of Euclidean geometry could be used to construct a model of *non-Euclidean geometry*, that is, a model satisfying all of Euclid's axioms *except* the parallel axiom.

Subsequently Dedekind (1872) discovered the real number continuum, and showed that the resulting Cartesian coordinate plane was now indeed a model of Euclid's axioms. This model gave rise, via the work of Beltrami, Klein and Poincaré, to models of non-Euclidean geometry as well. Both were revealed to be fully valid mathematical systems. Altogether these discoveries showed that the truth or falsity of the Parallel Postulate is not a mathematical question at all. It is an empirical question about the nature of physical space – a question that was not answered until the early twenty first century. Working in a different direction Riemann, Jordan and Lebegue developed a theory of measure that made it possible to specify which regions could be assigned an area or a volume and to describe methods to compute them.

4. Hilbert's Geometry

Drawing on these foundational advances in mathematics, the German mathematician David Hilbert first resolved the problems of Euclid's system in his 1899 *Grundlagen der Geometrie* [13]. This work provided a new system of axioms for plane geometry from which followed all of the necessary foundational facts and theorems together with a full theory of measurement of lengths, areas and angles. From Hilbert's now standard view of geometry, "point", "line", "between" and "congruent" are taken to be primitive (undefined) terms. They only acquire a meaning in a particular model of geometry, such as physical space or the Cartesian coordinate plane. Euclid's unstated assumptions about incidence and betweenness were fully justified by several new axioms. Adding Dedekind's completeness axiom allowed Hilbert to justify Euclid's unstated assumptions about intersections.

David Hilbert

Hilbert's system was quickly recognized by professional mathematicians as a fully satisfactory *Mathematical Foundation* for plane geometry. The first of many fine expositions of Hilbert's system was George Halsted's 1904 text *Rational Geometry: A Text-Book for the Science of Space* [12]. Colleges and universities adopted Hilbert's system as a prototype of modern mathematics, thereby fulfilling our *Axiomatic* goal. Moore developed his own version of Hilbert's axioms that has been widely used by his descendants for a *Guided Inquiry* course.

High schools had a quite different response to Hilbert's geometry. It took considerable time after the publication of Hilbert's work for the problems with Euclid to be fully recognized at the high school level. The 1913 edition of the widely used *Plane and Solid Geometry* by G. Wentworth and D.E. Smith [21] still adhered to the gentler approach of Euclid and made no attempt to incorporate Hilbert's system. In the preface of the 1960 geometry text of Brumfiel, Eicholz and Shanks [2], the authors refer to Hilbert's rectification of the omissions in Euclid:

> *"Means to remedy these gaps have been known for about sixty years, but strangely enough a mathematically adequate and yet elementary treatment of plane geometry in the spirit of Euclid has not appeared in print. This text represents an earnest effort to do just this."*

Subsequent experience showed that this text erred on the side of "*mathematically adequate*" rather than "*elementary*", and found a very limited following at the high school level. As educators became fully aware of the abstract and sophisticated nature of Hilbert's system, they agreed that it was not appropriate for a high school audience and continued to use Euclid.

5. Birkhoff's Geometry

George Birkhoff

In 1932 George Birkhoff published a quite different revision of plane geometry that he intended as a replacement for Euclid in the high schools [1]. To do this he drew on Dedekind's description of the real number continuum with two new axioms. The *ruler axiom* said that every line was order isomorphic to the real number line. The *protractor axiom* said that the angles off one side of a ray are order isomorphic to the real numbers between 0 and 180. From these two powerful axioms, together with a number of standard axioms, plane geometry smoothly flowed. The two new axioms drew on prior knowledge of the number system to gain ready access to the facts of geometry without the extended efforts required by Hilbert. For example, Hilbert's definition of a right angle as an angle congruent to its supplement requires an extended effort to prove that all right angles are congruent. In contrast, this follows immediately from Birkhoff's definition of a right angle as an angle with measure 90°.

It was not until the launch of Sputnik in 1957 that American educators acted on Birkhoff's proposal. The high school geometry component of the SMSG's New Math program finally dropped Euclid's formulation of plane geometry and adopted Birkhoff's system in its place. To this day, a version of Birkhoff's axioms has remained the backbone of high school geometry in the US and beyond.

6. Geometry in Turmoil

Hilbert and Birkhoff each offered a system of axioms from which they could prove all of Euclid's theorems and unstated assumptions, thereby fulfilling the *Foundations* goal. However, as we will see, neither Euclid, Hilbert nor Birkhoff provided the basic needs of a geometry curriculum for either high schools or universities.

As described above, Euclid's attempt to build an *Axiomatic Development* of geometry was his great contribution to western thought. It fell short of the *Foundations* goal only because it required foundational parts of mathematics that were not discovered until much later. In spite of these omissions, Euclid's geometry was a key inspiration for many people up to the late 1970s. It provided the content we still look for in geometry today, including congruence, area measure, angle measure, similarity and properties of circles. It fulfilled the *Guided Inquiry* goal as the forerunner of modern guided inquiry learning. And it certainly served to awaken my own interest in mathematics, as a tenth grade geometry student in 1962, despite its shortcomings.

As we have seen, Hilbert's geometry is too advanced for high school students. At the university level it nicely fulfills the *Axiomatic* and *Guided Inquiry* goals. From his small set of elegantly parsimonious axioms all of the necessary foundational facts can be proven. But a rigorous development of axiomatic geometry from Hilbert's axioms is a lengthy and sophisticated process requiring considerable time and a serious commitment to abstract mathematics. This rigorous detail comes with a price because university students can not normally devote more than one semester to geometry. The time required means that a one semester Hilbert geometry course will miss many of the topics that are standard parts of the high school curriculum.

Unfortunately the use of Euclid in high schools and Hilbert in colleges has led to a serious conflict. Geometry is particularly important for university students preparing for K-12 teaching who typically make up a sizable portion of university enrollments in geometry. Felix Klein complained that these students were being subjected to a cruel "double discontinuity" in which there was practically no connection between the range of topics they learned and later taught in high school and the geometry of Hilbert they studied while at the university. In his article, "The Mis-Education of Mathematics Teachers" [22], Wu likened this situation to one in which we prepare pre-service French teachers by teaching them Latin at the university and then asking them to make the necessary adjustments when their own students arrive to learn French.

The point Klein and Wu both made is that, before we do anything else, our primary obligation to pre-service teachers is to provide them with a deep and thorough knowledge of the topics they will actually need to teach. This is surely sound advice for every discipline, and I have added it here as a final goal for a successful university geometry course.

- *Standard Content.* Give university students an in depth understanding of the standard topics that are taught in high school geometry.

The teaching of Euclid's geometry in high school while teaching Hilbert's geometry at universities was an example of a serious violation of this *Content* requirement that challenged us to find other options.

Since Euclid's geometry was abandoned in our schools, high school geometry has had recurrent difficulties. Varying policies for handling it have been tried and then met with frustration over this period, so far without any broad agreement as to what should be done. But there are two fundamental features of these policies that have not changed during this time. It is my belief that the real problem lies with those two features, and that a satisfactory resolution will not come without freeing ourselves of both.

The first feature is the high stakes exams and their consequences. This concern applies not only to geometry, but to most school subjects. We have seen growing pressure for accountability of our schools. The idea is that educators should get together and say in advance what they expect students to be able to do as a result of being taught and how they will document students' ability to meet those expectations. This documentation is normally acquired through state administered exams. Superficially the request for this documentation from the tax payers who finance public education seems reasonable.

The difficulty with this request is that **there is a very poor correlation between what is important for students to learn and what can be reliably assessed on a state exam.** As the stakes get higher, educators are pressed to shift what they teach away from what students need most to learn in favor of what is more readily documented on an exam. In geometry we would like our students to be able to think creatively, solve problems and prove theorems on their own, and effectively articulate their results. But these skills are not easy to assess on large scale exams. Much easier to assess is content knowledge of piecemeal facts and procedures that are of much less value to the students. Ultimately this discrepancy brings us to a choice between

(i) teaching what students need most to learn and assessing their knowledge as best we can, or

(ii) teaching what we can most readily assess and slanting its content toward student needs as best we can.

As is the case with many other subjects, we have in geometry chosen (ii) over (i). I support the Common Core State Standards as a promising effort to implement (i) in place of (ii). As I see it the problem with the CCSS is not the standards themselves, but rather the unrealistic requirement that their long term benefits be assessed with a state exam. (See James Loewen's similar analysis of the teaching of history [17].)

The second feature of high school geometry that has not changed in the past thirty five years is Birkhoff's axiom system. The "advantage" of this system is that the powerful ruler and protractor axioms give high school students quick access to the many facts of geometry. But in practice this has made it ideally suited for the current data intensive curriculum that efficient assessment demands. Furthermore this quick access comes at cost of a serious additional liability.

Birkhoff's premise was that students could avoid the lengthy task of building geometry from Hilbert's parsimonious axioms by drawing on their prior knowledge of the real number system. However, we must ask, just what prior knowledge of the real number system does a young high school student have? The real number continuum is a complex system that took centuries to fully understand, and is normally studied in upper division real analysis courses at a university. As such it is well beyond anything that can be viewed as prior knowledge of high school students. Indeed it is hard to see that high school students understand *anything* about the real numbers beyond their prior knowledge of the rational numbers. The existence of a number representing the length of the diagonal of a unit square or the ratio of circumference to diameter of a circle can only be justified by the authority of their book or their teacher.

The extent to which Birkhoff's geometry depends on the real number system is hard to over estimate. Imagine we have a plane that satisfies Birkhoff's axioms. Choose any line in that plane, and coordinatize it using the Ruler Axiom. Now take a second line that is perpendicular to the first, intersecting the first at 0. Coordinatize the second line with 0 at the intersection. Given real numbers x and y, we can construct a perpendicular to the first line at the point with coordinate x and to the second at the point with coordinate y. The parallel axiom says that these two perpendiculars will cross at some point P. To P we can assign the coordinates (x, y). It quickly follows at the outset that our plane is isomorphic to the coordinate plane $\mathbb{R}^2$. Consequently Birkhoff's geometry is easily seen to reduce to a study of coordinate geometry.

This conclusion about Birkhoff's geometry is not just an abstract fact. It has had a very significant impact on high school geometry from the time the program was first introduced. The transition to Birkhoff geometry saw the name of the course change from "Geometry" to "Integrated Mathematics". Since there was only one model of the axioms, the axioms themselves had limited relevance. The one model $\mathbb{R}^2$ could be studied by combining geometry and algebra as a single course sequence. When properties of the real numbers were needed, they were declared to be true by fiat. Immersing students in a concurrent study of algebra, a subject new to them as well, left them even more dependent on books, teachers and calculators to know what was true. For example, they can offer no evidence whatsoever that the properties of numbers, like the distributive law, hold beyond the rational numbers.

Birkhoff's geometry does cover the *Content* of high school geometry, as it gives ready access to the required facts. But it fails the *Axiomatic* goal because, in practice, the role of the axioms is barely visible to students. It also fails the *Guided Inquiry* goal by relying so heavily on unsubstantiated facts about real numbers. Altogether it begs the question as to why we teach geometry at all.

The accompanying summary table shows which of the four goals are met by each of the three historically used plane geometry curricula. As we see, it confronts us with a compelling dilemma. How do we construct a geometry curriculum that will provide an axiomatic setting for guided inquiry learning (as Birkhoff could not do), supply a sound mathematical foundation (as Euclid could not do) and at the same time cover the standard topics of plane geometry (as Hilbert could not do)?

Summary Table	Euclid	Hilbert	Birkhoff
Axiomatic Development	$\ast$	$\ast$	
Guided Inquiry Learning	$\ast$	$\ast$	
Mathematical Foundations		$\ast$	$\ast$
Standard Content	$\ast$		$\ast$

7. Resolution

After years of teaching Hilbert's geometry [3] to prospective teachers, a point came when I suddenly discovered that they were no longer getting an axiomatic development of Euclid's geometry in high school. This explained their difficulty with Hilbert, and brought home to me Klein's double discontinuity. Determined to find a better alternative for them, I obtained a grant from the Educational Advancement Foundation to write the text [4] that would reach the four goals I have described. In this section I will outline the strategy I used to do this.

As is normally done, I assume foundational knowledge of naive set theory and logic without mentioning it explicitly. Students use the logic and properties of sets that seem right to them, and instructors or other students raise discussion about anything that seems questionable. These practices are common place in mathematics classrooms and have functioned well for a long time. To this naive foundational knowledge of set theory and logic, I add the following three *foundational principles*. My intention is that these principles be handled just as the the principles of naive set theory and logic are handled. (See Appendix C of [4] for a list of Hilbert's axioms, and any text that uses them, such as [3], [11] or [14], for consequences of Axioms 1 to 7.)

Incidence. *Consequences of Hilbert's Axioms 1, 2 and 3 can be used.*

Betweenness. *Consequences of Hilbert's Axioms 4, 5, 6 and 7 can be used.*

Intersections. *Intersections which appear to always exist can be used.*

Multiple pilot tests of this text by many instructors have shown this strategy to be very effective. Students will make informal use of these foundational principles as they make of naive set theory and logic. If a question ever arises, we state the relevant principle explicitly and reassure them that they can use it. Not needing to establish or cite these principles as part of the axiomatic structure saves a great deal of time, time that can be spent covering the required content. A brief look at the table of contents of [4] shows that it does indeed satisfy the *Content* goal, covering the standard topics of synthetic geometry as outlined in the Common Core State Standards.

The text [4] is written in a *Guided Inquiry* format. It includes neither proofs of theorems nor worked examples. Instead, it consists of only the necessary explanations and definitions together with a carefully designed sequence of problems and theorems that are to be solved and proven by the students. They are organized in gentle but challenging steps, starting with a small set of axioms, so that each item can be done by drawing on what came before it. As a guided inquiry study, I have minimized what I ask students to accept on my authority. There are 200 problems and theorems listed, 197 for the students to do on their own. The remaining 3 are theorems whose proofs require arguments that would be a bit of a digression to include. They are instead presented without proof: the Rectangle Area Theorem, the Scaling Theorem and the Theorem Pi. For each of these 3 the students will prove approximating cases to get a sense of why they might be true.

I use *Axiomatic Development* based on ten axioms, each presented in the text at the point that it is first needed. Axioms 2, 3 and 5 are familiar: SSS, SAS and the Euclidean Parallel Axiom. A very weak but essential Axiom 4 says that if a ray emanates from the vertex of an angle and is otherwise interior to the angle, then that angle is not congruent to either of the two smaller angles formed by the ray. Axiom 8, saying that dilations preserve betweenness, leads to the standard theorems about triangle similarity. The remaining three axioms about plane geometry, Axioms 1, 6 and 7, all use a similar format to describe measure: length measure, area measure and angle measure. In the last chapter Axioms 9 and 10 on solid geometry lead to the well known principles of perspective drawing. Experience has shown that these axioms are all fully understandable for students and are consistent with their beliefs about physical space.

It is not at all apparent from [4] itself that it satisfies the *Foundations* goal. To fulfill this goal we need a mathematical basis for the foundational principles and for the three unproven theorems, as was not available to Euclid. For the Incidence and Betweenness Principles, the cited axioms of Hilbert give us this. The remainder has been provided by Samrat Pathania in [19]. He restates the Intersections Principle as the consequences of a small set of axioms asserting the existence of specific intersections. He then shows that the results of [4] can all be proven with these additional axioms, including the Rectangle Area Theorem, Scaling Theorem and Theorem Pi.

8. Conclusion

High school mathematics has three primary themes: algebra, functions and geometry. Under these themes students acquire essential tools to understand any of the vast range of quantitative components of the modern world. Today many career directions depend in an essential way on a mastery of these tools.

High school algebra is about the algebra of the real numbers. Functions studied in high school are functions from real numbers to real numbers. As we have observed here, high school students have very limited knowledge of the real number system. From prior years they have a sound knowledge of the rational number system, but knowledge of the real numbers beyond the rationals comes from real analysis – an advanced university subject available only to those majoring in mathematics. For high school students, knowledge of the real number system must necessarily be transmitted through the authority of teachers and textbooks. There is simply no alternative. This means that the goals of understanding an *Axiomatic Development* and constructing mathematics through *Guided Inquiry* are inherently limited in a high school study of algebra and functions.

This shows us why the study of geometry in high school was always viewed as so essential in the past. It was the one remaining opportunity to meet these two goals at the high school level. The replacement of Euclid's geometry with Birkhoff's geometry squandered this opportunity, as Birkhoff's geometry depends on the real number system just as much as high school algebra and functions do. My central point is that it is now time to correct this error.

It is my hope that the text [4] will bring many students a recognition that they have the ability to solve problems and prove theorems on their own, without needing to cite authorities. I hope that teachers will get from it a better idea of what mathematics is about and why we need them to teach it. And perhaps, some day, the formulation of geometry presented in [4] will find its way, through some appropriate reformulation, into our high schools so that students will no longer wonder why they are being told to study geometry.

References

[1] G. D. Birkhoff, "A Set of Postulates for Plane Geometry (Based on Scale and Protractors)", Annals of Mathematics 33 (1932), 329345.

[2] C. F. Brumfiel, R. E. Eicholz, M. E. Shanks, *Geometry*, Addison-Wesley, 1960.

[3] D. Clark, "Hilbert Geometry", *Journal of Inquiry-Based Learning in Mathematics*, ⟨www.jiblm.org⟩, Issue #22, 2011.

[4] D. Clark, *Euclidean Geometry: A Guided Inquiry Approach*, AMS and MSRI: Math Circles Library #9, (2012).

[5] C. A. Coppin, W. T. Mahavier, E. L. May, G. E. Parker, *The Moore Method: A Pathway to Learner-Centered Instruction*, Notes Series MAA, 2009.

[6] L. Deslauriers, E. Schelew, C. Wieman, "Improved Learning in a Large-Enrollment Physics Class", *Science*, Vol. 332, May 2011, 862–864.

[7] Euclid of Alexandria, *Elements of Geometry*, 300 BC. (English translation: T. J. Heath, *The Thirteen Books of Euclid's Elements*, Cambridge: Cambridge University Press, 1908.)

[8] H. Eves, *An Introduction to the History of Mathematics*, Saunders, 1990.

[9] S. Freeman et. al., "Active learning increases student performance in science, engineering, and mathematics", *Proceedings of the National Academy of Sciences*, Vol. 111, No. 23, (2014), 8410-8415.

[10] P. Freire, *Pedagogy of the Oppressed*, 30th Anniversary ed., New York: Continuum Pub. Co., 2006.

[11] M. Greenberg, *Euclidean and Non-Euclidean Geometries*, W. H. Freeman and Company, 1974.

[12] G. B. Halsted, *Rational Geometry: A Text-Book for the Science of Space*, New York: John Wiley, 1904.

[13] D. Hilbert, *Grundlagen der Geometrie*, 1899. (English translation: E. J. Townsend, *The Foundations of Geometry*, Illinois: Open Court Pub. Co, 1902.)

[14] R. F. Jolly, *Synthetic Geometry*, Holt, Rinehart and Winston, 1969.

[15] M.Kogan and S.Laursen, "Assessing Long-Term Effects of Inquiry-Based Learning: A Case Study from College Mathematics", Innov. High. Educ. 39, 2014.

[16] S. Laursen, M. Hassi, M. Kogan, A Hunter, T. Weston, *Evaluation of the IBL Mathematics Project: Student and Instructor Outcomes of Inquiry-Based Learning in College Mathematics*, Ethnography and Evaluation Research Center, U. of Colorado, 2011. (Preprint of 203 page report.)

[17] J. W. Loewen, *Lies My Teacher Told Me: Everything Your American History Textbook Got Wrong*, Simon and Schuster, 2007.

[18] J. Parker, *R. L. Moore, Mathematician and Teacher*, The Mathematical Association of America, 2005.

[19] S. Pathania, "Foundations of MCL #9 Geometry", M.S. Thesis, in preparation.

[20] A. Spencer, "On Attracting and Retaining Mathematics majors—Don't Cancel the Human Factor", *Notices of the American Mathematical Society*, Vol. 42, No. 8, August 1995, 859–862.

[21] G. Wentworth, D. E. Smith, *Plane and Solid Geometry*, Ginn and Co., 1913.

[22] H. Wu, "The Mis-Education of Mathematics Teachers", *Notices of the American Mathematical Society*, Vol. 58, No. 3, March 2011, 372–384.

Your comments and feedback are welcome!

David Clark ⟨*clarkd@newpaltz.edu*⟩